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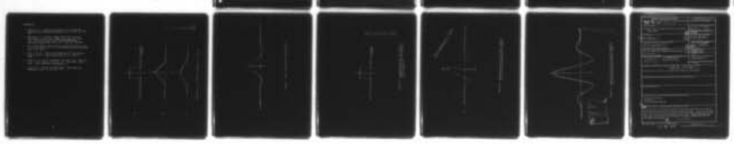

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OPTIMAL PREFILTERING FOR SPLINE RECONSTRUCTION

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ABSTRACT

The use of an optimal prefilter prior to sampling can lead to reduced mean square error after reconstruction. The form of the best prefilter or the associated local weighting function depends upon the reconstruction method to be used. We display and discuss these weighting functions for the most common spline reconstruction methods.

This report extends the results of "Minimal Error Sampling", University of Maryland Computer Science Center Technical Report TR-380. The support of the Directorate of Mathematical and Information Sciences, U. S. Air Force Office of Scientific Research, under Contract F44620-72C-0062, is gratefully acknowledged, as is the help of Shelly Rowe. The author wishes to thank Dr. A. Rosenfeld for his encouragement and assistance.

The purpose of this note is to indicate that the use of the standard delta function filter for sampling is sub-optimal when standard reconstruction techniques such as nearest neighbor, linear, or cubic spline interpolation are used for signal approximation. The errors induced in reconstruction using these schemes have led to various suggested modifications of the reconstruction methods. Such modifications are often impractical because of their computational complexity. This note points out that mean square errors can be minimized by the use of a specific <u>pre</u>filter, i.e., by modifying the sampling technique.

The use of prefiltering to reduce reconstruction error is not new. For example, it is known that the (L_2-) optimal prefilter for delta sampling followed by (sinx)/x-reconstruction is a low pass filter [1]. In the case where the original signal is band-limited, exact reconstruction is possible. However, in precision processing of digitized images, for example, as well as in other signal processing applications, (sinx)/x interpolation is often considered too costly when interpolated values are needed for geometric correction and display magnification [2]. The usual interpolation schemes include nearest neighbor approximation, linear interpolation, and cubic spline interpolation, which we may regard as spline interpolation schemes of orders zero, one, and three respectively. The substitution of spline interpolation for sinx/x reconstruction calls for a new prefilter; despite this fact, most imaging systems

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attempt to sample using a pure low pass filter followed by delta-function sampling. Indeed, restoration theories attempt to correct for inaccuracies in this "ideal" sampling filter, even in the case of schemes which represent ideal reconstructions by cubic splines [3].

The optimal prefilter for a given spline reconstruction scheme is the Fourier transform of the optimal weighting function. We will determine the optimal weighting functions for the three aforementioned spline schemes assuming infinitely many equispaced knots (sampling points) in one dimension. The problem is equivalent to that of finding the best L_2 spline approximation to a given noise-free function with fixed knots [4]. By using the L_2 norm, we are led to a linear relationship between the sample values $\{y_i\}$ used for reconstruction and the initial signal $f(x)$. This relationship is represented by the weighting function g_i, i.e.,

$$y_i = \int_{-\infty}^{\infty} g_i(x) f(x) dx \tag{1}$$

We have assumed infinitely many sample points so as to assure that the g_i will all be translates of a single optimal weighting function g, which is the desired solution. Extensions to the cases of finitely many points, higher dimensions, and other linear reconstruction techniques are straightforward, based on the same standard numerical analysis approach used below, once the reconstruction method is properly formulated.

Denote by A the transformation which takes a sequence of sample values $\{y_i\}_{i=-\infty}^{\infty}$ into the interpolating spline (or reconstructed function) $s(x)$ with equispaced knots located at points of the sequence $\{x_i\}_{i=-\infty}^{\infty}$. Suppose that A restricted to square summable sequences ℓ_2 is a bounded linear transformation into $L_2(R)$. Then for a given function $f \in L_2$, the sequence $\vec{y} = \{y_i\}$ which minimizes $||A\vec{y} - f||_2$ can be shown to be a solution to the "normal equation"

$$(A^*A)\vec{y} = A^*f \tag{2}$$

where A^* is the adjoint transformation from $(L_2(R))^*$ to $(\ell_2)^*$, i.e., from $L_2(R)$ to ℓ_2. Provided A^*A is nonsingular, the optimal sample y_i is a projection of the linear operator $(A^*A)^{-1}A^*$ operating on f, and thus the optimal weighting function g_i is the L_2 dual element representing that functional. This establishes equation (1).

By energy conservation considerations, $\int_{-\infty}^{\infty} g_i(x)dx = 1$, and so in fact $g_i \in L_1(R)$. Thus $g_i(x)$ is the optimal weighting function for general $L_\infty(R)$ functions for minimizing the <u>uniformly averaged</u> mean square error. Finally, if A is covarient with respect to rigid left and right shifts of the samples $y_i \to y_{i+k}$ represented by $s(x) \to s(x+x_k)$, then the weighting functions g_i are independent of i modulo interval shifts, thus yielding a unique optimal weighting function g(x).

For spline reconstruction, we have

$$s(x) = A\vec{y} = \sum_{i=-\infty}^{\infty} y_i \phi_i(x), \tag{3}$$

where the $\phi_i(x)$ are basis spline functions chosen to suit the particular application. If we regard A as an infinite matrix whose kth column is the function $\phi_k(x)$, $-\infty < k < \infty$, and the variable x represents the row number, then $A^* = A^T$, and the i,j entry of A*A is the value $\int_{-\infty}^{\infty} \phi_i(x)\phi_j(x)dx$. The desired weighting function g_i is then simply the ith row of the matrix $(A^*A)^{-1}A^*$, which is the spline generated by the values $\{b_{ik}\}_{k=-\infty}^{\infty}$, where $(b_{ij}) = (A^*A)^{-1}$.

It is customary to use B-spline basis functions for the $\phi_i(x)$, which are shifts of the B-spline basis functions at zero (Figure 1). Since the B-splines are symmetric with finite support, the resulting infinite matrix (A*A) is banded with constant bands. For zero order splines, A*A is simply the identity matrix. For linear splines, A*A is infinite tridiagonal, with entry 1/6 in the codiagonals, and 2/3 in the diagonal. For cubic splines, A*A has seven nonzero bands, whose values, incidentally, correspond to $B_7(x_i)$, i = -3, -2, -1, 0, 1, 2, 3, where B_7 is the B-spline of order 7 centered at x_0.

In each case, A*A is invertible, so that we can solve for the optimal samples $\{y_i\}$ used for reconstruction. However, for the cubic spline case, a special adjustment is needed. Because the cubic B-spline has support three intervals wide, the value of the reconstructed spline at the sample points is given by the formula

$$s(x_i) = (1/6)y_{i-1} + (2/3)y_i + (1/6)y_{i+1}.$$

Although these values $\{y_i\}$ are useful for constructing $s(x)$, since the cubic B-spline is easily computed, our usual notion of sampling and interpolation requires that the sample y_i satisfy $s(x_i) = y_i$. Indeed, y_i should be close to $f(x_i)$, but not necessarily equal due to the prefiltering. This latter condition on the $\{y_i\}$ is satisfied by the values

$$\vec{y} = C(A^*A)^{-1}A^*f,$$

where (4)

$$C = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & 1/6 & 2/3 & 1/6 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

for A obtained from the cubic B-spline. The weighting functions are obtained from the rows of $C(A^*A)^{-1}A^*$. The spline $s(x)$ can be reconstructed from these modified samples using the unique <u>bounded</u> cubic basis splines $\Psi_i(x)$ satisfying $\Psi_i(x) = \delta_{ij}$, from the formula $s(x) = \sum_{-\infty}^{\infty} \Psi_i(x)$. For infinitely many equispaced points, the Ψ_i are all translates of Ψ_0, shown in Figure 2. Even though the Ψ_i have infinite support, they decay rapidly, so that the sum, to machine precision, is finite. In fact, we could use the Ψ_i's to define A, in which case the adjustment using the matrix C is unnecessary. However, using a cubic B-spline to define A assures that A*A will be easy to compute and will be banded.

In each case, the appropriate operators are covariant with respect to shifts, so that we obtain unique optimal weighting functions, shown in Figure 3. As expected, for nearest neighbor approximation the optimal samples are obtained from local unweighted block averages. The optimal weighting functions for linear and cubic spline interpolation are more surprising, despite the fact that an optimal prefilter is a familiar concept. The functions are not spline approximations to (sinx)/x, since they decay exponentially. The oscillatory behavior results from inverting the positive matrix A*A, and is governed more by the convergence of the partial quotients of a continued fraction than by the function (sinx)/x [5]. Because the functions in Figure 3 decay so rapidly, truncated versions with a support radius of two or three intervals yield very nearly the same optimal samples, which is often a distinct advantage over (sinx)/x weighting.

In any case, the observation that the weighting functions behave qualitatively like (sinx)/x is consistent with the earlier result that (sinx)/x reconstruction calls for low-pass prefiltering -- i.e., a (sinx)/x weighting function. Specifically, if we use basis functions

$$\Psi_i(x) = \frac{\sin(\pi/h)(x-x_i)}{(\pi/h)(x-x_i)}$$

to define A in equation 3, then we may solve (2) to obtain the classical weighting function for Shannon reconstruction. Since $(\mathrm{sinc} * \mathrm{sinc})(x) = \pi\,\mathrm{sinc}(x)$, a fact which follows from the Fourier equivalent $\mathrm{rect}^2(x) = \mathrm{rect}(x)$, we obtain $(A^*A) = hI$, I the identity matrix. So the rows of $(A^*A)^{-1}A^*$ are all translates of the classical low-pass weighting function

$$g(x) = \frac{1}{h} \cdot \frac{\sin(\pi/h)x}{(\pi/h)x} \quad .$$

Error analysis shows that substantial reduction in expected mean square error can be achieved by using the appropriate prefilter from Figure 3, as opposed to unfiltered delta sampling. For example, if the autocorrelation function of the initial signal decays exponentially, and the sampling is very fine, a 35% reduction in total expected mean square error can be obtained by using optimal sampling prior to linear reconstruction, as opposed to normal delta sampling [5].

Implementation of these results involves construction of the optimal sampling weighting functions for local integration, or their Fourier transforms for spectral filtering. If one is resampling finely sampled data to achieve compression, and expects to use spline reconstruction, then appropriate weights must be assigned to the values in the neighborhood of the new sampling point. One can obtain these weights by resolving the normal equations for quantized functions f, or by using approximations from the continuous case.

To construct the linear weighting function of Figure 3b, one need only find the linear spline interpolating the values of the central row of $(A^*A)^{-1}$. Thus one must invert an infinite tridiagonal matrix, which is an easy procedure using the recursion formula:

$$b_0 = \frac{1}{c_1\sqrt{(c_0/c_1)^2-4}}$$

$$b_1 = \frac{1}{2c_1} - \frac{c_0}{2c_1} b_0,$$

$$b_{k+1} = -(c_0/c_1)b_k - b_{k-1},$$

where c_0 is the entry in the diagonal, and c_1 is the off diagonal entry. In this case, $c_0 = 2/3$, and $c_1 = 1/6$. The middle row of the inverse matrix is $\{\ldots,b_{-1},b_0,b_1,\ldots\}$. Because of overflow difficulties, in general one first solves for $z_k = b_k/b_{k+1}$ when calculating the b_i's [6]. Incidentally, constructing the cubic basis function of Figure 2 involves the same matrix inversion to obtain second moments at the knots.

For the optimal sampling cubic weighting function (Figure 3c), an infinite banded matrix with seven nonzero bands must be inverted. Since the matrix is diagonally dominant, finite versions of the matrix are easily inverted. These, as it turns out, converge rapidly to the desired infinite version as the size is increased, due to the rapid decay of the inverse matrix values. A more general inversion technique for this case would certainly be welcomed. In both cases, however, to reconstruct the appropriate spline weighting function, one need only store the few significant nonzero values $(b_0, b_1, b_2, b_3, \ldots)$ corresponding to $g(x_0)$, $g(x_1), \ldots, g(x_3), \ldots$. Since g is the appropriate spline interpolating these values, it can be obtained easily from spline interpolation routines and a short table of previously computed values for $g(x_i)$.

To summarize, we are recommending the substitution of one of the weighting functions of Figure 3 for the standard delta function sampling method, whenever practical. Our analysis and construction of these weighting functions assumed the case of infinitely many equispaced samples, which permitted the presentation of the theory of optimal prefiltering using the classical Banach spaces L_2 and ℓ_2. Since the resulting weighting functions decay rapidly, they provide very nearly optimal samples for the finite case everywhere except within two or three sample points of the endpoints. In addition, their effectively narrow width permits easy direct implementation of the local weighted sum for sampling. Extension of this theory to sampling in several dimensions is straightforward, especially if the reconstruction basis functions are separable in each variable. Finally, the reconstructed function may be considerably closer to the original function in the mean square sense if one of the optimal prefilters is applied prior to sampling.

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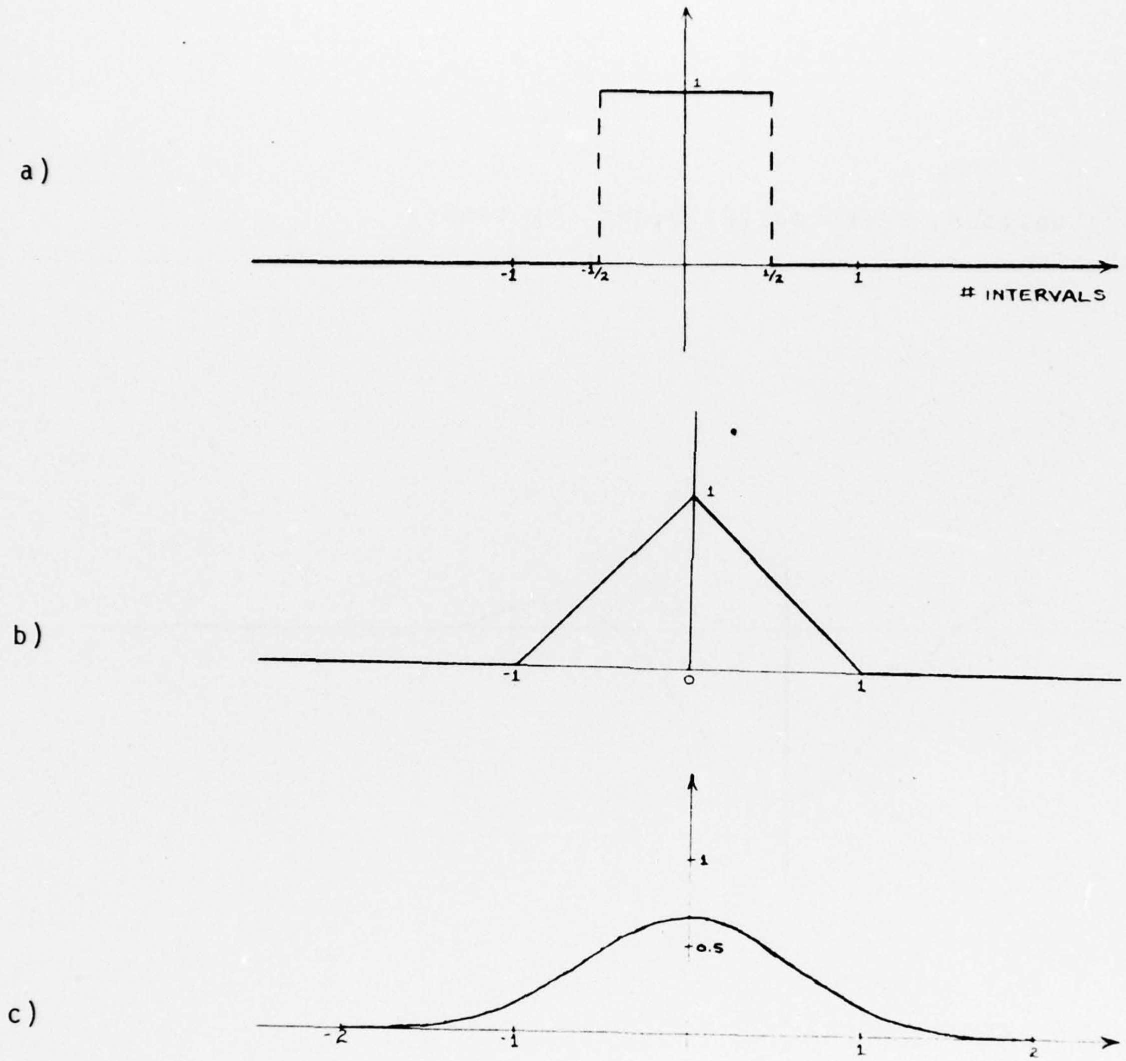

Figure la-c. B-splines of degrees 0, 1, and 3.

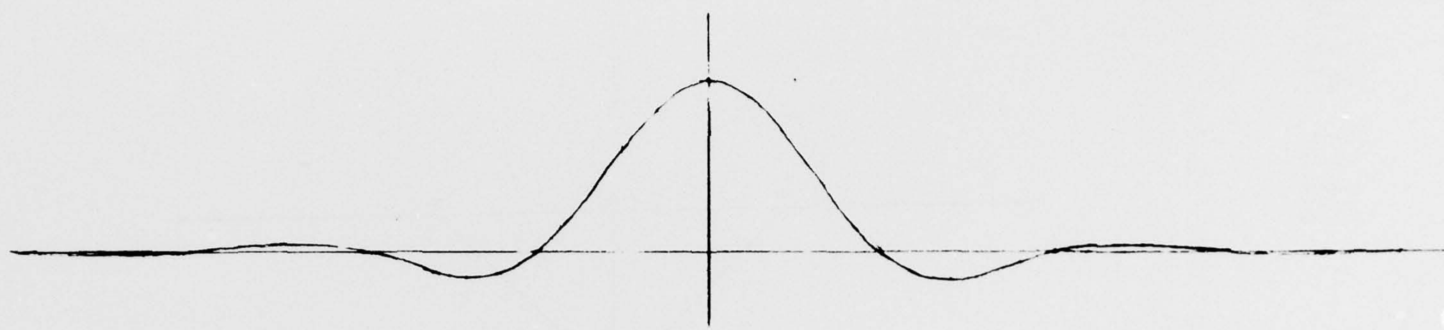

Figure 2. Cubic spline basis function $\Psi_0(x)$.

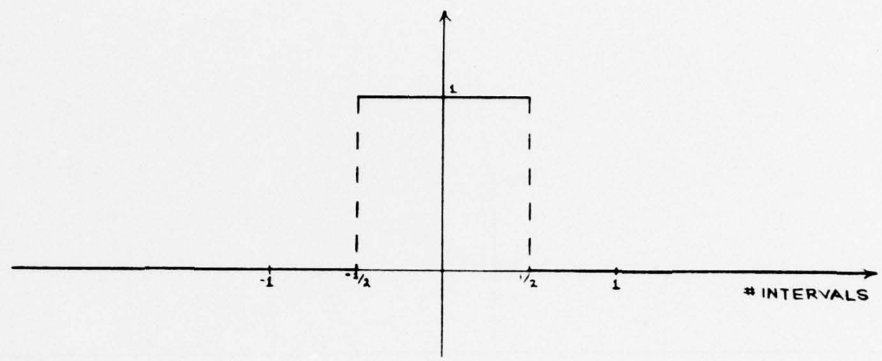

Figure 3a. Weighting function for optimal sampling using spline reconstruction of order 0.

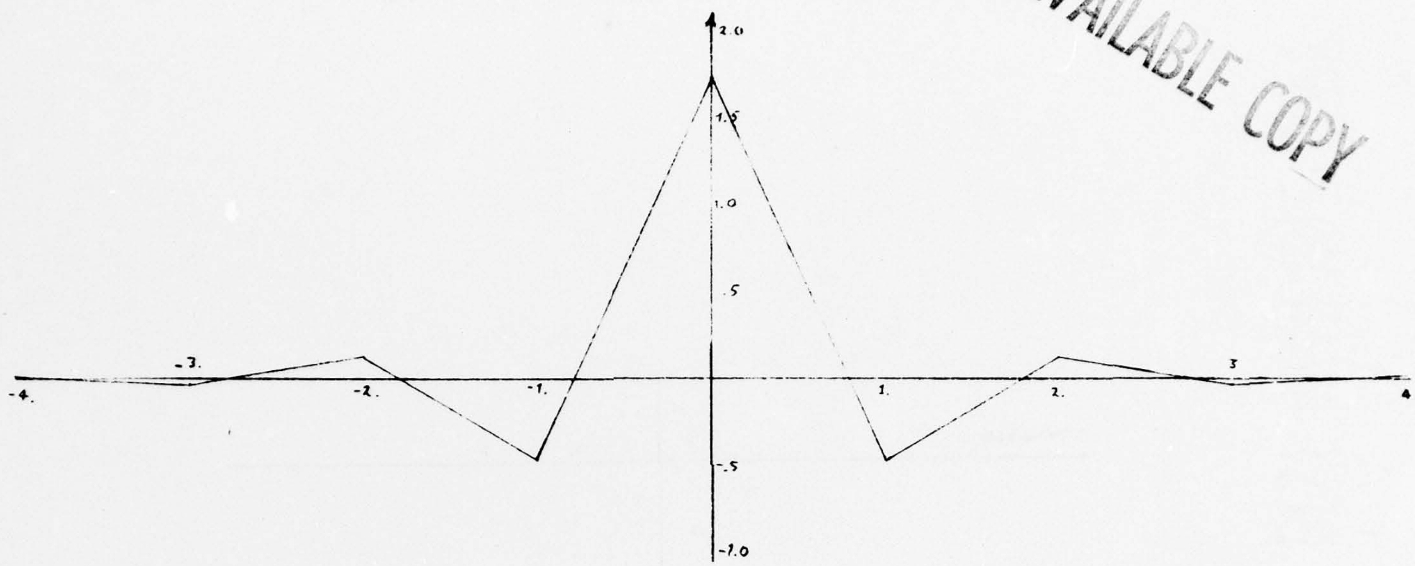

Figure 3b. Weighting function for optimal sampling using spline reconstruction of order 1.

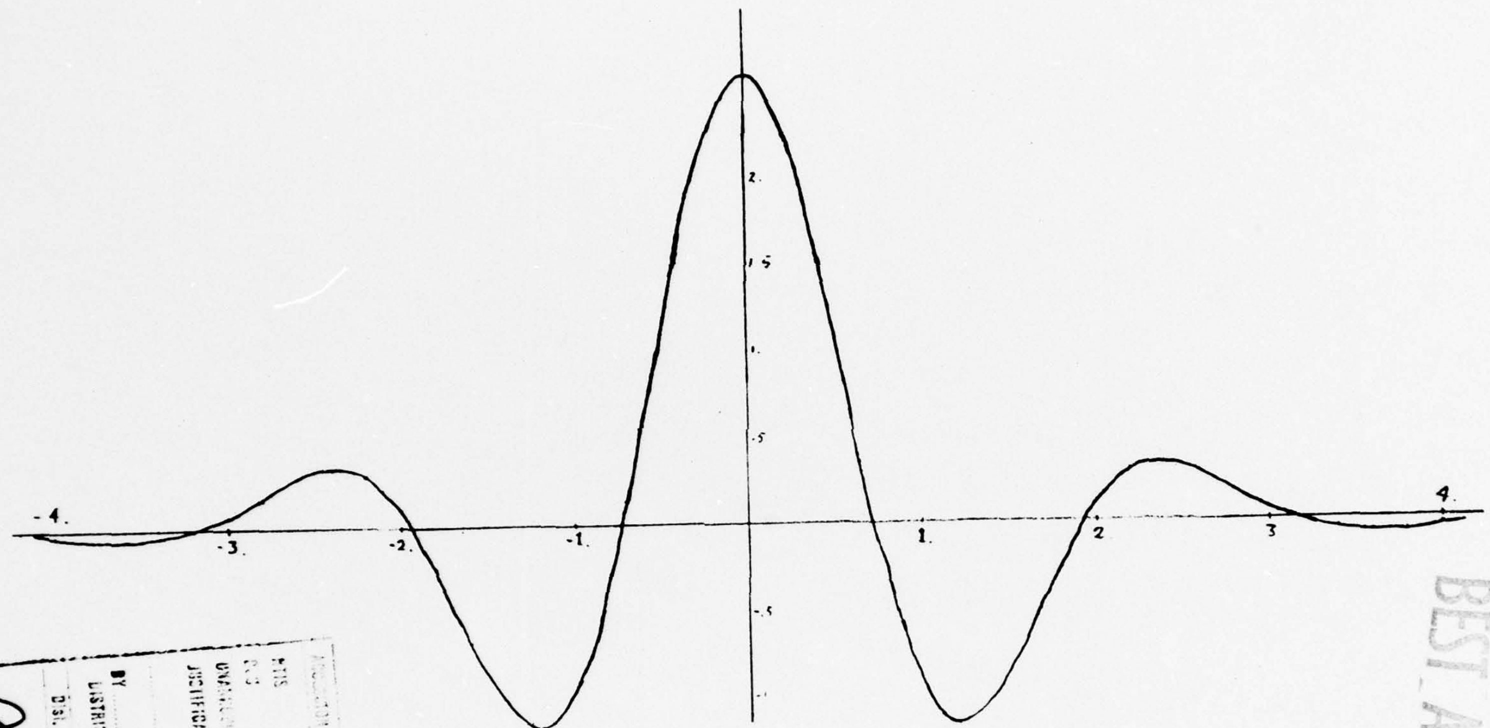

Figure 3c. Weighting function for optimal sampling using spline reconstruction of order 3.

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